Ginsparg-Wilson Relation and Admissibility Condition in Noncommutative Geometry*)

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Ginsparg-Wilson relation and admissibility condition have the key role to construct lattice chiral gauge theories. They are also useful to define the chiral structure in finite noncommutative geometries or matrix models. We discuss their usefulness briefly.

§1. Introduction

Noncommutative (NC) geometry[1] had attracted much attention recently from various motivations. Topologically nontrivial configurations in finite NC geometries or matrix models[2][3] have been constructed based on algebraic K-theory and projective modules in many papers, but it would be better if we could obtain an index operator which takes an integer even at a finite cutoff, since we need to perform the Kaluza-Klein compactification of extra dimensions with nontrivial indices to construct four dimensional chiral gauge theories. This can be realized if we utilize the Ginsparg-Wilson (GW) relation[4] and the admissibility condition[5][6],[7] which were developed in lattice gauge theory (LGT) to construct chiral gauge theories.[8]

§2. GW formulation in finite NC geometry

In ref.,[9] we proposed a general prescription to construct chirality and Dirac operators satisfying the GW relation and an index in general gauge field backgrounds on general finite NC geometries. The prescription proposed in ref.[9] is as follows. Let us introduce two hermitian chirality operators: one is a chirality operator γ, which is assumed to be independent of gauge fields, while the other is constructed in terms of a hermitian operator H as $\hat{\gamma} \equiv \frac{H}{\sqrt{H^2}}$, $H^\dagger = H$. γ and $\hat{\gamma}$ satisfy $\gamma^2 = \hat{\gamma}^2 = 1$. $\hat{\gamma}$ depends on gauge fields through H. The Dirac operator D_{GW} is defined by $1 - \gamma\hat{\gamma} = f(a,\gamma) D_{GW}$, where a is a small parameter. H and the function f must be properly chosen so that the D_{GW} is free of species doubling and behaves correctly in the commutative limit ($a \to 0$). D_{GW} satisfies the GW relation:[4] $\gamma D_{GW} + D_{GW}\hat{\gamma} = 0$. Therefore the fermionic action $S_F = \mathrm{tr}(\bar{\Psi} D_{GW} \Psi)$ is invariant under the modified chiral transformation[10),11),9] $\delta\Psi = i\lambda\hat{\gamma}\Psi$, $\delta\bar{\Psi} = i\bar{\Psi}\lambda\gamma$. The Jacobian, however, is not invariant and has the form $q(\lambda) = \frac{1}{2}\mathcal{T}r(\lambda\hat{\gamma} + \lambda\gamma)$, where $\mathcal{T}r$ is a trace of operators

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acting on matrices. This $q(\lambda)$ is expected to provide a topological charge density, and the index for $\lambda = 1$.

An index theorem is given by index$D_{GW} \equiv (n_+ - n_-) = \frac{1}{2}\mathcal{T}r(\gamma + \hat{\gamma})$, where $n_\pm$ are numbers of zero eigenstates of D_{GW} with a positive (or negative) chirality (for either γ or $\hat{\gamma}$). This index theorem can be easily proven,[12] as done in LGT[13].[10] The index is invariant under small deformation of any parameters such as gauge configurations in the operator H. We note that $\hat{\gamma}$ becomes singular when H has zero modes. When an eigenvalue of H crosses zero, the value of $\mathcal{T}r\hat{\gamma}$ changes by two.

In LGT the configuration space of gauge fields is topologically trivial if we do not impose an admissibility condition[5),6),7)] on gauge fields. This condition suppresses the fluctuation of gauge fields, and consequently forms a topological structure composed of isolated islands in the configuration space. This condition also excludes zero modes of H. In ref.[9] we have thus expected that a similar mechanism would work also in finite NC geometries or matrix models, and that the index could take various integers according to gauge configurations.

§3. The index on fuzzy 2-sphere

In ref.[9] we provided a set of simplest chirality and Dirac operators on fuzzy 2-sphere, as a concrete example given by the prescription. The set in the absence of gauge fields corresponds to that constructed earlier in ref.[14] The properties of D_{GW} and other types of Dirac operators D_{WW}[15] and D_{GKP}[16] are summarized in Table I, which suggests that some kind of Nielsen-Ninomiya's theorem exists in matrix model or NC geometry. The properties of these Dirac operators are also discussed in ref.[17),14)] D_{WW} has no chiral anomaly. The source of the chiral anomaly in D_{GKP} is the breaking in a cut-off scale of the action under the chiral transformation,[18] and that in D_{GW} is the Jacobian. The nontrivial Jacobian is shown to have the correct form of the Chern character in the commutative limit.[9]

Table I. The properties of three types of Dirac operators on fuzzy 2-sphere

Dirac operator	chiral symmetry		no doublers	counterpart in LGT
D_{WW}	$D_{WW}\Gamma + \Gamma D_{WW} = 0$	○	×	naive fermion
D_{GKP}	$D_{GKP}\Gamma + \Gamma D_{GKP} = \mathcal{O}(1/L)$	×	○	Wilson fermion
D_{GW}	$D_{GW}\hat{\Gamma} + \Gamma D_{GW} = 0$	○	○	GW fermion

D_{GW} works well. The index, however, cannot take nonzero integers on fuzzy 2-sphere. We need to apply projective modules to the index so that it can take nonzero integers[19].[12] The modified index is symbolically expressed as index$D_{GW} = \frac{1}{2}\mathcal{T}r\left\{P^{(m)}[A_\mu^{(m)}](\gamma + \hat{\gamma}[A_\mu^{(m)}])\right\} = m$. The gauge fields $A_\mu^{(m)}$ are determined dependent on m. $P^{(m)}$ is a projector to pick up a Hilbert space on which $\hat{\gamma}$ acts. The insertion of $P^{(m)}$ is necessary on fuzzy 2-sphere. The configuration with $m = \pm 1$[19] is interpreted as the 't Hooft-Polyakov monopole[12)20)].[21] As explained above, the index cannot take nonzero integers on fuzzy 2-sphere without the projector. Furthermore, the naive imposition of an admissibility condition on gauge fields, which

can be written down so that zero modes of H are excluded, results in providing just a vacuum sector with trivial configurations. On a NC torus, however, the situation changes, since gauge fields on a NC torus are defined compactly as in LGT.

§4. The index on a NC torus

Since a NC torus[22)] has a lattice structure,[23)] we can use the overlap Dirac operator,[24)] which is a practical solution to the GW relation in LGT, by replacing lattice difference operators with their matrix correspondences on the NC torus.[25)] We can also construct it by the prescription explained in section 2.[26)] The nontrivial Jacobian on the NC torus is shown to have the form of the Chern character with star-products in a weak coupling expansion[26)] by utilizing a topological argument in ref.[27)] Parity anomaly is also calculated in ref.[28)]

On the NC torus the gauge action is given by $S_G = N\beta \sum_{\mu>\nu} \mathrm{tr}\left[1 - \frac{1}{2}(P_{\mu\nu} + P_{\mu\nu}^\dagger)\right]$ where $P_{\mu\nu}$ is the plaquette. Its explicit representation is given in ref.[29),26)] This is the TEK model[30),31)] which was shown to be a nonperturbative description of NC Yang-Mills theory[32),23)] In ref.[29)] we formulated an admissibility condition on a NC torus. The admissibility condition is given by $\|1 - P_{\mu\nu}\| < \eta_{\mu\nu} \quad \text{for all } \mu > \nu$, where $\eta_{\mu\nu}$ are some positive parameters. Applying arguments in refs.[6),7)] onto the NC torus, it is shown that zero modes of H are excluded if we choose $\eta_{\mu\nu}$ properly. The admissibility condition implies $\|\,[\nabla_\mu, \nabla_\nu]\,\| < \eta_{\mu\nu}/a^2$, which is the bound on the field strength. This becomes irrelevant in the continuum limit. In this sence this condition is natural.

The index can be calculated by evaluating the eigenvalues of H. Namely, the index is equal to half of the difference of the number of the positive eigenvalues of H and that of the negative ones. In ref.[29)] generating many configurations of U_μ which satisfies the admissibility condition, we numerically analyzed the index on the simplest $d = 2$ dimensional NC torus, and found various configurations with non-trivial indices. Since the index is topologically invariant against small deformation of configurations, this result shows that a topological structure is naturally realized in the gauge field space by the admissibility condition, and that the index can take nonzero integers without utilizing projective modules on a NC torus.

§5. Discussions

GW relation and admissibility condition have an essential role in finite NC geometries or matrix models as well as in LGT. It is important to construct and investigate[21)] GW fermions on various NC geometries according to the prescription.[9)] It is also important to study in detail the index[33)] on a NC torus to analyze the validity of the admissibility condition proposed in ref.[29)] We hope to report some progress in these directions in the future.

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